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SPACES OF COMPACT OPERATORS AND THEIR DUAL SPACES

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ABSTRACT
Let E and F be reflexive Banach spaces and $\mathscr{C}(E,F)$ the space of all compact linear operators from E to F. A representation of the dual space of $\mathscr{C}(E,F)$ is given and it is proved that $\mathscr{C}(E,F)$ is either reflexive or nonconjugate. Applications of these results are also given.

1. Introduction

In 1957 R. Schatten [11] proved that, if H is an infinite dimensional Hilbert space, the space $\mathscr{C}(H,H)$ of all compact linear transformations from H to H equipped with the sup norm is not a conjugate space. Let E and F be Banach spaces and $\mathscr{C}(E,F)$ the space of all compact linear transformations from E to F equipped with the sup norm. It is natural to ask if it is possible to generalize the result of Schatten. We know that, in special cases, $\mathscr{C}(E,F)$ is reflexive. (For example, if $E = l^p$, $F = l^q$, $1 < q < p < +\infty$, cf. J. R. Holub [6].) In Section 4 of this paper the following result is proved.

THEOREM. *Let E and F be reflexive Banach spaces. Then, $\mathscr{C}(E,F)$ is either reflexive or nonconjugate.*

To obtain this result, we give in Section 3, a representation of the dual space of $\mathscr{C}(E,F)$ as a quotient space of the completion of the tensor product $F^*\otimes E^{**}$ equipped with the projective norm π, in some special cases.

2. Preliminaries and notations

1) All the Banach spaces that shall be considered are over the same field K, the real or the complex field. The Banach spaces are equipped in general with the norm topology. Let E be a Banach space, E^* the conjugate space of E. We denote by $\sigma(E,E^*)$ the weak topology on E defined by the duality between E and E^*. One says E is a conjugate space if there exists a Banach space X such that $X^* = E$, as Banach spaces.

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2) Let E and F be Banach spaces. We denote by $\mathscr{L}(E,F)$ the space of all linear continuous transformations, or "operators", from E to F, $\mathscr{C}(E,F)$ the space of all compact, linear transformations from E to F, $\mathscr{R}(E,F)$ the space of all compact linear transformations from E to F which are limits in the norm topology of linear continuous transformations of finite rank. All these spaces equipped with the norm of operators are Banach spaces. The norm of an element x of E is denoted by $\|x\|$; the norm of an element T of $\mathscr{L}(E,F)$ is denoted by $\|T\|$. We denote by T^* the transposed transformation of T from F^* to E^* and by T^{**} the transformation $(T^*)^*$. We say that a linear continuous transformation T from E to F is a quotient map if T is onto and if $E/\ker T$ and F are canonically isometric.

3) Let E be a Banach space; we set:

$$U(E) = \{x \mid x \in E \quad \text{and} \quad \|x\| \leqq 1\}$$

$$\Sigma(E) = \{x \mid x \in E \quad \text{and} \quad \|x\| = 1\}.$$

For all $\eta > 0$ and $x \in E$ we denote

$$B(x,\eta) = \{y \mid y \in E \qquad \text{and} \quad \|x - y\| < \eta).$$

Let E be a real Banach space and M a convex subset of E. A point x of M is called a strongly exposed point of M if there exists $x^* \in E^*$ such that: for every $\varepsilon > 0$, there exists $\delta > 0$, such that for all $z \in M \setminus B(x,\varepsilon)$ we have $\langle x - z, x^*\rangle \geqq \delta$. In this case, we say that x is strongly exposed by x^*. One sees easily that if x is a strongly exposed point of $U(E)$ (by x^*) then $\langle x, x^*\rangle = \|x^*\|$. From J. Lindenstrauss [9] and S. Troyanski [14] (see also M. Day [1]) one knows that in a Banach space every convex weakly compact set is the closed convex hull of its set of strongly exposed points.

4) Let u be an element of $E \otimes F$,

$$u = \sum_{i=1}^{n} x_i \otimes y_i, \qquad x_i \in E, \quad y_i \in F. \tag{1}$$

We define $\|u\|_\pi = \inf \Sigma_i \|x_i\| \, \|y_i\|$, the infimum being taken over all the representations of u of the form (1). One knows that π is a $\otimes$ norm (cf. [5] or [10] for the definition of a $\otimes$ norm). We denote by $E \hat{\otimes}_\pi F$ the completion of $E \otimes F$ equipped with π. If $u \in E \hat{\otimes}_\pi F$ there is a representation (cf. A. Grothendieck [4] or R. Schatten [11])

$$u = \sum_{i=1}^{\infty} x_i \otimes y_i \quad \text{with} \quad \sum_{i=1}^{\infty} \|x_i\| \, \|y_i\| < +\infty. \tag{2}$$

Moreover, $\|u\|_\pi = \inf \Sigma_i \|x_i\| \, \|y_i\|$, the infimum being taken over all the representations of u of the form (2). Let F_1 be another Banach space and let $T \in \mathscr{L}(F, F_1)$. We define $Tu = \Sigma_{i=1}^\infty x_i \otimes Ty_i$; the element $Tu \in E \hat{\otimes}_\pi F_1$ and $\|Tu\|_\pi \leqq \|T\| \, \|u\|_\pi$. There exists a natural transformation from $E^* \hat{\otimes}_\pi F$ to $\mathscr{L}(E, F)$. If $T \in \mathscr{L}(E, F)$ is in the image of this transformation, one says that T is nuclear. The space of nuclear transformations from E to F is denoted by $N_1(E, F)$. The space $N_1(E, F)$ is a quotient space of $E^* \hat{\otimes}_\pi F$. If E^* or F has the approximation property (see [5]), one knows that $E^* \hat{\otimes}_\pi F$. and $N_1(E, F)$ can be identified as Banach spaces.

5) Let $u \in E^* \hat{\otimes}_\pi E$, $u = \Sigma_{i=1}^\infty x_i^* \otimes x_i$ with $\Sigma_{i=1}^\infty \|x_i\| \, \|x_i^*\| < +\infty$. The functional $u \to \mathrm{tr}(u) = \Sigma_{i=1}^\infty \langle x_i, x_i^* \rangle$ is well defined, linear and of norm less than or equal to 1 (cf. [4]). R. Schatten [12] showed that$(E \hat{\otimes}_\pi F)^* = \mathscr{L}(F, E^*)$. If $u \in E \hat{\otimes}_\pi F$ and $T \in \mathscr{L}(F, E^*)$, the duality between $E \hat{\otimes}_\pi F$ and $\mathscr{L}(F, E^*)$ is expressed by the formula: $\langle u, T \rangle = \mathrm{tr}(Tu)$.

6) Let ε be the $\otimes$ norm on $E \otimes F$ induced by $\mathscr{L}(E^*, F)$ and $E \hat{\otimes}_\varepsilon F$ the completion of $E \otimes F$ equipped with ε. Let $I_1(F, E^*)$ be the Banach space of all integral operators from F to E^* (cf. [4] or [10]). A. Grothendieck [4] showed that $(E \hat{\otimes}_\varepsilon F)^* = I_1(F, E^*)$.

7) In general, $N_1(F, E^*) \subset I_1(F, E^*)$. One knows (cf. J. Diestell [2] or C. Stegall [13]) that if F^* (resp E^*) has the Radon-Nikodym Property (RNP), then for every Banach space E (resp F) $N_1(F, E^*) = I_1(F, E^*)$ as linear topological spaces (see also the remark following Theorem 1). In particular, it is the case if E or F is reflexive.

3. On the dual of $\mathscr{C}(E, F)$

THEOREM 1. *Let E and F be Banach spaces such that E^{**} or F^* has the RNP and $\mathscr{C}(E, F)$ the space of all compact operators from E to F. Let $v \in F^* \hat{\otimes}_\pi E^{**}$, $T \in \mathscr{C}(E, F)$, V the transformation from $F^* \hat{\otimes}_\pi E^{**}$ to $(\mathscr{C}(E, F))^*$ defined by $\langle T, V(v) \rangle = \mathrm{tr}(T^{**}v)$. Then*:

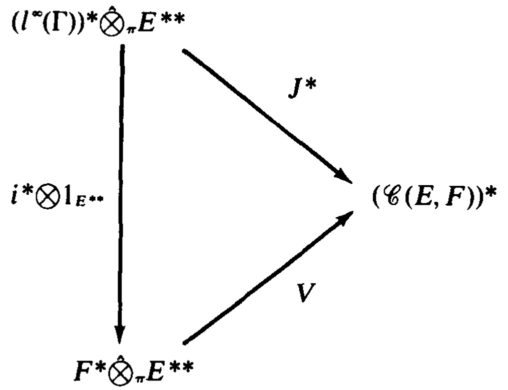
1) *The transformation V is a quotient map. Moreover, for all $\phi \in (\mathscr{C}(E, F))^*$, there exists $v \in F^* \hat{\otimes}_\pi E^{**}$, such that $\phi = V(v)$* and $\|\phi\| = \|v\|_\pi$.

2) *If B_1 is the canonical transformation from $F^* \hat{\otimes}_\pi E^{**}$ onto $I_1(F, E^{**})$, then* $\ker V \subset \ker B_1$.

PROOF. Let $v \in F^* \hat{\otimes}_\pi E^{**}$. Then the transformation $T \to \mathrm{tr}(T^{**}v)$ from $\mathscr{C}(E, F)$ to the scalar field is linear and continuous. Let V be the transformation from $F^* \hat{\otimes}_\pi E^{**}$ to $(\mathscr{C}(E, F))^*$ defined by $\langle T, V(v) \rangle = \mathrm{tr}(T^{**}v)$. It is clear that V is linear and continuous and of norm less than or equal to 1.

a) Suppose that E^{**} has the RNP. Let Γ be $U(F^*)$, i the canonical injection of F into $l^\infty(\Gamma)$ and J the isometry from $\mathscr{C}(E,F)$ to $E^*\hat{\otimes}_\varepsilon l^\infty(\Gamma)$ defined for all $T\in\mathscr{C}(E,F)$, by $J(T)=iT$.

Since E^{**} has the RNP and $(l^\infty(\Gamma))^*$ the metric approximation property (cf. A. Grothendieck [5]), it is clear that $I_1(l^\infty(\Gamma),E^{**})=(l^\infty(\Gamma))^*\hat{\otimes}_\pi E^{**}$, as Banach spaces. Then we have the following diagram:



We shall show that this diagram is commutative:

Let $\mu\in(l^\infty(\Gamma))^*$, $x^{**}\in E^{**}$, $T\in\mathscr{C}(E,F)$. Then:

$$\begin{aligned}\langle T, J^*(\mu\otimes x^{**})\rangle &= \langle J(T),\mu\otimes x^{**}\rangle\\ &= \langle iT,\mu\otimes x^{**}\rangle\\ &= \langle (iT)^*(\mu), x^{**}\rangle\\ &= \langle \mu,(iT)^{**}(x^{**})\rangle.\end{aligned}$$

On the other hand:

$$\begin{aligned}(i^*\otimes 1_{E^{**}})(\mu\otimes x^{**}) &= i^*(\mu)\otimes x^{**}.\\ \langle T, V(i^*(\mu)\otimes x^{**})\rangle &= \operatorname{tr}(T^{**}\circ(i^*(\mu)\otimes x^{**}))\\ &= \langle i^*(\mu), T^{**}x^{**}\rangle\\ &= \langle \mu,(iT)^{**}x^{**}\rangle.\end{aligned}$$

Let $v=\mu\otimes x^{**}$. We see that:

$$J^*(v)=(V(i^*\otimes 1_{E^{**}}))(v).$$

It follows that the same is true for all $v \in (l^\infty(\Gamma))^*\hat{\otimes}_\pi E^{**}$. Hence the diagram is commutative. But J is an isometry. Then J^* is surjective and so is V. Let $\phi \in (\mathscr{C}(E,F))^*$. There exists $u \in (l^\infty(\Gamma))^*\hat{\otimes}_\pi E^{**}$ such that, for all $T \in \mathscr{C}(E,F)$,

$$\begin{aligned}\phi(T) &= \mathrm{tr}(uiT)\\ &= \langle J(T), u\rangle\\ &= \langle T, J^*u\rangle,\end{aligned}$$

and:

$$\|\phi\| = \|u\|_\pi .$$

Let $u = (i^*\otimes 1_{E^{**}})(u)$. We have:

$$V(v) = \phi$$

and:

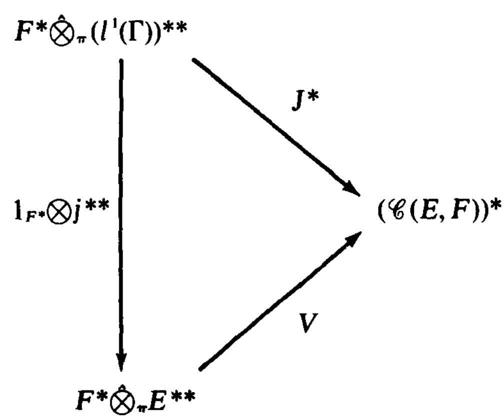
$$\|\phi\| \leqq \|v\|_\pi \leqq \|u\|_\pi = \|\phi\|.$$

Thus, $\|\phi\| = \|v\|_\pi$. Result 1 is obtained for this case.

b) Suppose now that F^* has the RNP. We know that there exist a set Γ_1 and a canonical quotient map j of $l^1(\Gamma_1)$ onto E. Let J be the isometry from $\mathscr{C}(E,F)$ to $(l^1(\Gamma_1))^*\hat{\otimes}_\pi F$ defined by $J(T) = Tj$ for all $T \in \mathscr{C}(E,F)$. Since F^* has the RNP and $(l^1(\Gamma_1))^{**}$ the metric approximation property (cf. A. Grothendieck [5]), it follows that:

$$I_1(F, l^1(\Gamma_1))^{**}) = F^*\hat{\otimes}_\pi (l^1(\Gamma_1))^{**}$$

as Banach spaces. We have the following diagram:



One shows easily that this diagram is commutative. Now it is possible to obtain result 1 as in part a). We now have to prove result 2. Let B_1 be the canonical transformation from $F^*\hat{\otimes}_\pi E^{**}$ to $I_1(F, E^{**})$. Since E^{**} or F^* has the RNP, we know that $I_1(F, E^{**})$ is isomorphic to $(F^*\hat{\otimes}_\pi E^{**})/\ker B_1$. But $(E^*\hat{\otimes}_\varepsilon F)^* = I_1(F, E^{**})$ and also $E^*\hat{\otimes}_\varepsilon F \subset \mathscr{C}(E, F)$. It follows immediately from the first part that $\ker V \subset \ker B_1$.

REMARK 1. By using the method of Theorem 1 we can prove the following: Let E and F be Banach spaces such that E^* or F^* has the RNP. Then $N_1(F, E^*) = I_1(F, E^*)$ as Banach spaces. Moreover, if B_1 is the canonical map from $F^*\hat{\otimes}_\pi E^*$ onto $I_1(F, E^*)$, then for all $T \in I_1(F, E^*)$, there exists $u \in F^*\hat{\otimes}_\pi E$ such that

$$B_1(u) = T \quad \text{and}$$

$$\|u\|_\pi = i_1(T), \text{ where } i_1(T) \text{ is the integral norm of } T$$

(cf. [4] or [10]).

COROLLARY 1.1. *If E^{**} or F^* has the RNP, $(\mathscr{C}(E, F)^{**}$ can be identified isometrically with a subspace of $\mathscr{L}(E^{**}, F^{**})$. By this identification, the canonical transformation from $\mathscr{C}(E, F)$ to $(\mathscr{C}(E, F))^{**}$ maps T to T^{**}.*

PROOF. The transformation V is a quotient map. Then we can identify $(\mathscr{C}(E, F))^{**}$ with a subspace of $\mathscr{L}(E^{**}, F^{**})$ by the transformation V^* which is an isometry. Let J_1 be the transformation $T \to T^{**}$ from $\mathscr{C}(E, F)$ to $\mathscr{L}(E^{**}, F^{**})$ and J_2 the canonical transformation from $\mathscr{C}(E, F)$ to $(\mathscr{C}(E, F))^{**}$.

To conclude the proof, it is sufficient to prove that $J_1 = V^* J_2$.

Let $v \in F^*\hat{\otimes}_\pi E^{**}$ and $T \in \mathscr{C}(E, F)$. We have:

$$\langle T, V(v)\rangle = \langle V(v), J_2(T)\rangle = \langle v, V^*J_2(T)\rangle.$$

But

$$\langle T, V(v)\rangle = \operatorname{tr}(T^{**}v) = \langle v, J_1(T)\rangle.$$

The result follows.

Corollaries 1.2 and 1.3 are both generalizations and special cases of results of D. R. Lewis [8] and N. J. Kalton [17].

COROLLARY 1.2. *Suppose that E^{**} or F^* has the RNP and let (T_α) be a bounded net in $\mathscr{C}(E, F)$. Then the following conditions are equivalent:*

1) *T_α convergence to 0 weakly (that is to say, for the topology $\sigma(\mathscr{C}(E, F), (\mathscr{C}(E, F))^*)$,*

2) *For all $x^{**} \in E^{**}$ and $y^* \in F^*$, $\langle y^*, T_\alpha^{**}x^{**}\rangle$ converges to 0.*

PROOF. 1) $\Rightarrow$ 2). Suppose that 1) is true. Let $x^{**} \in E^{**}$ and $y^* \in F^*$. The functional $T \to \langle y^*, T^{**}x^{**} \rangle$ on $\mathscr{C}(E,F)$ is linear and continuous. Hence 2) follows.

2) $\Rightarrow$ 1). Let $\phi \in (\mathscr{C}(E,F))^*$. There exists $u \in F^* \hat{\otimes}_\pi E^{**}$ such that $\phi(T_\alpha) = \mathrm{tr}(T_\alpha^{**}u)$. It follows from 2) by a standard argument that $\mathrm{tr}(T_\alpha^{**}u)$ converges to 0.

COROLLARY 1.3. *Let E and F be reflexive Banach spaces, m a positive real number and* (T_n) *a sequence in* $\mathscr{C}(E,F)$ *such that* $\| T_n \| \leqq m$. *Then, there exists a subsequence* (T_{n_k}) *of* (T_n) *such that* (T_{n_k}) *is a weak Cauchy sequence.*

PROOF. There exists a subsequence (T_{n_k}) such that for all $x \in E$ and $y^* \in F^*$, $\langle T_{n_k}x, y^* \rangle$ is a convergent sequence (adapt the method of Y. Gordon, D. R. Lewis and J. R. Retherford [15], p. 115, for this case). Let $S_k = T_{n_k}$. From an observation of Pełczyński, it is sufficient to prove that for all increasing subsequences of indices k_i, $S_{k_{i+1}} - S_{k_i}$ converges to 0 weakly. It is clear from Corollary 1.2.

From Theorem 1, it is also possible to obtain the following result of Holub [7]:

COROLLARY 1.4. *Let E and F be reflexive Banach spaces. If* $\mathscr{L}(E,F) = \mathscr{C}(E,F)$, *then* $\mathscr{L}(E,F)$ *is reflexive.*

PROOF. If E and F are reflexive, we have from Corollary 2, $\mathscr{C}(E,F) \subset (\mathscr{C}(E,F))^{**} \subset \mathscr{L}(E,F)$. Since the canonical transformation from $\mathscr{C}(E,F)$ to $(\mathscr{C}(E,F))^{**}$ is the transformation $T \to T$ (Corollary 2), the result follows.

REMARK 2. The result obtained in Theorem 1 is in a certain sense best possible: If E^{**}, for example, does not have the RNP, we shall show that there exists a Banach space F such that the map V is not surjective. In this case, one knows by C. Stegall [12] that there exists a Banach space F such that $N_1(F, E^{**}) \neq I_1(F, E^{**})$. Let T be an integral operator from F to E^{**} which is not nuclear, K be the unit ball of F^* equipped with the weak * topology, i the cannonical map from F to the Banach space of continuous functions on K, $C(K)$. We know (cf. A. Grothendieck [4]) that there exists a positive measure μ on K such that if $L^1(K,\mu)$ denotes the Banach space of classes of functions defined on K, with scalar values, integrable with respect to μ, we have the following factorization of T:

$$T : F \xrightarrow{i} C(K) \xrightarrow{j} L^1(K,\mu) \xrightarrow{U} E^{**},$$

j being the canonical map from $C(K)$ to $L^1(K,\mu)$ and U linear and continuous. It is clear that Uj is integral and not nuclear. Then $N_1(C(K),E^{**}) \neq I_1(C(K),E^{**})$. Take here $F = C(K)$. It is clear that $\mathscr{C}(E,C(K)) = E^*\hat{\otimes}_\varepsilon C(K)$. Then, $(\mathscr{C}(E,C(K)))^* = I_1(C(K),E^{**})$. The map V from $(C(K))^*\hat{\otimes}_\pi E^{**}$ to $(\mathscr{C}(E,C(K)))^*$ is here the canonical map from $(C(K))^*\hat{\otimes}_\pi E^{**}$ to $I_1(C(K),E^{**})$. Since $N_1(C(K),E^{**}) \neq I_1(C(K),E^{**})$, this map is not surjective.

It is possible to obtain an analogous result if F^* fails to have the RNP.

REMARK 3. Generally, if E and F are reflexive Banach spaces then $(\mathscr{C}(E,F))^{**} \subset \mathscr{L}(E,F)$. If also E or F has the approximation property, then equality holds, i.e., $(\mathscr{C}(E,F))^{**} = \mathscr{L}(E,F)$. It is natural to ask if equality must always be true even when both E and F fail to have the approximation property. We now give a counter-example. Let E be the subspace of $l^p (p > 2)$ demonstrated by F. E. Alexander [16] to have the following properties:

1°. E fails to have the approximation property;

2°. $\mathscr{R}(E,E) \neq \mathscr{C}(E,E)$,

and, as shown by A. M. Davie (oral communication):

3°. There does not exist a bounded net (T_α) of compact operators such that $T_\alpha x \to x$ for every x in E.

Now, let B_1 and V be the maps defined in Theorem 1 where $F = E$.

By 1° it follows that $\ker B_1 \neq (0)$. Also, by 2° it follows that $\ker V \neq \ker B_1$. By 3° we shall see that $(\mathscr{C}(E,E))^{**} \neq \mathscr{L}(E,E)$ and thus $\ker V \neq (0)$.

Suppose, to the contrary, that $(\mathscr{C}(E,E))^{**} = \mathscr{L}(E,E)$, then U_1, the unit ball of $\mathscr{C}(E,E)$, is dense in U_2, the unit ball of $\mathscr{L}(E,E)$, for the weak* topology $\sigma(\mathscr{L}(E,E), E^*\hat{\otimes}_\pi E)$. Since E is reflexive, it follows that U_1 is dense in U_2 for the weak operator topology. By [3], p. 477, Corollary 5, it follows that U_1 is dense in U_2 for the strong operator topology, in contradiction to 3°. Hence 3° implies $(\mathscr{C}(E,E))^{**} \neq \mathscr{L}(E,E)$. It follows that $\ker V \neq (0)$.

To resume, we have for this space E,

$$(0) \neq \ker V \underset{\neq}{\subset} \ker B_1.$$

4. Spaces of compact operators

THEOREM 2. *Let E and F be reflexive Banach spaces and G a closed linear subspace of $\mathscr{C}(E,F)$ which contains $\mathscr{R}(E,F)$. Then G is either reflexive or nonconjugate.*

PROOF. Since E is reflexive we know by Theorem 1 that there exists a canonical quotient map V from $F^*\hat{\otimes}_\pi E$ onto $(\mathscr{C}(E,F))^*$. Let J be the inclusion map from G to $\mathscr{C}(E,F)$ and $\alpha = J^*V$. It is clear that α is also a quotient map. Suppose that G is the conjugate of a Banach space X. Then we may consider X as a subspace of $X^{**} = G^*$. We shall consider two cases:

Case a) The scalar field is real.

Let x and y^* be strongly exposed points of $U(E)$ and $U(F^*)$ respectively. Suppose the points x and y^* are exposed by x^* and y respectively and we may assume that $\|x^*\| = \|y\| = 1$. One knows (cf. Preliminaries, part 3) that:

$$\|x\| = \|x^*\| = \|y\| = \|y^*\| = \langle x, x^*\rangle = \langle y, y^*\rangle = 1.$$

Let ε be a positive number. Then there exists a real number δ, $0 < \delta < 1$, such that:

$$\text{For all} \quad z \in U(E)\backslash B(x,\varepsilon), \qquad \langle x - z, x^*\rangle \geqq \delta \quad \text{and}$$

$$\text{For all} \quad z^* \in U(F^*)\backslash B(y^*,\varepsilon), \qquad \langle y, y^* - z^*\rangle \geqq \delta.$$

The element $x^*\otimes y$ is in $G = X^*$ and $\|x^*\otimes y\|_\varepsilon = 1$. Hence there exists $v \in X$ such that $\|v\| < 1 + \min(\varepsilon\delta, 1)$ and $\langle v, x^*\otimes y\rangle = 1$. The transformation α is a quotient map; thus there exists $u \in F^*\hat{\otimes}_\pi E$ such that $\alpha(u) = v$ and $\|u\|_\pi < 1 + \min(\varepsilon\delta, 1)$.

There is a representation of u of the form $u = \sum_{i=1}^{\infty}\lambda_i y_i^*\otimes x_i$, such that for all i, $\lambda_i \geqq 0$, $\|x_i\| = \|y_i^*\| = 1$ and $\sum_i\lambda_i < 1 + \min(\varepsilon\delta, 1)$.
We may also assume that $\langle x_i, x^*\rangle \geqq 0$, for all i. We define the following subsets of the set of the natural numbers N:

$$I = \{i \mid i \in N,\ \|x - x_i\| < \varepsilon \ \text{and} \|y^* - y_i^*\| < \varepsilon\},$$

$$L = N\backslash I.$$

Since

$$1 = \langle v, x^*\otimes y\rangle = \langle \alpha(u), x^*\otimes y\rangle,$$

we have

$$(*) \qquad 1 = \sum_{i=1}^{\infty} \lambda_i \ \langle x_i, x^*\rangle \langle y, y_i^*\rangle.$$

Then:

$$1 \leqq \sum_{i\in I} \lambda_i + (1-\delta)\sum_{i\in L} \lambda_i,$$

$$1 \leqq \sum_{i=1}^{\infty} \lambda_i - \delta \sum_{i\in L} \lambda_i.$$

Thus:

$$\sum_{i\in L}\lambda_i \leqq \frac{1}{\delta}\left(\sum_{i=1}^{\infty}\lambda_i - 1\right)$$

$$\sum_{i\in L}\lambda_i < \varepsilon.$$

Now we shall estimate $\|u - y^*\otimes x\|_\pi$:

$$\|u - y^*\otimes x\|_\pi = \left\|\sum_{i\in N}\lambda_i y_i^*\otimes x_i - y^*\otimes x\right\|_\pi$$

$$\leqq \left\|\sum_{i\in L}\lambda_i y_i^*\otimes x_i\right\|_\pi + \left\|\sum_{i\in I}\lambda_i y_i^*\otimes x_i - \sum_{i\in I}\lambda_i y^*\otimes x\right\|_\pi$$

$$+\left\|\sum_{i\in I}\lambda_i y^*\otimes x - y^*\otimes x\right\|_\pi.$$

But,

$$\left\|\sum_{i\in L}\lambda_i y_i^*\otimes x_i\right\|_\pi \leqq \sum_{i\in L}\lambda_i < \varepsilon.$$

If $i \in I$, we have:

$$\|y_i^*\otimes x_i - y^*\otimes x\|_\pi \leqq \|y_i^*\otimes x_i - y_i^*\otimes x\|_\pi + \|y_i^*\otimes x - y^*\otimes x\|_\pi$$

$$< 2\varepsilon.$$

Thus:

$$\left\|\sum_{i\in I}\lambda_i y_i^*\otimes x_i - \sum_{i\in I}\lambda_i y^*\otimes x\right\|_\pi < 2\varepsilon\sum_{i\in I}\lambda_i < 4\varepsilon.$$

Also:

$$\left\|\sum_{i\in I}\lambda_i y^*\otimes x - y^*\otimes x\right\|_\pi = \left|\sum_{i\in I}\lambda_i - 1\right|.$$

But:

$$-\varepsilon < 1 - \sum_{i\in L}\lambda_i - 1 \leqq \sum_{i=1}^{\infty}\lambda_i - \sum_{i\in L}\lambda_i - 1$$

$$= \sum_{i\in I}\lambda_i - 1 \leqq \sum_{i=1}^{\infty}\lambda_i - 1 < \varepsilon.$$

Hence:

$$\left|\sum_{i\in I}\lambda_i - 1\right| < \varepsilon.$$

Thus:

$$\|u - y^*\otimes x\|_\pi \leqq \varepsilon + 4\varepsilon + \varepsilon$$

$$\leqq 6\varepsilon.$$

But $\alpha^{-1}(x)$ is closed. And it follows that $y^*\otimes x \in \alpha^{-1}(X)$.

Let x be a strongly exposed point of $U(E)$ and $y^*_0 \in U(F^*)$. From the results of Lindenstrauss and Troyanski (cf. Preliminaries, part 3) it is clear that we also have $y^*_0\otimes x \in \alpha^{-1}(X)$.

Let $x_0 \in U(E)$ and $y^*_0 \in U(F^*)$. By the same reasoning we obtain $y^*_0\otimes x_0 \in \alpha^{-1}(X)$.

Then, it follows that $F^*\hat{\otimes}_\pi E = \alpha^{-1}(X)$ or:

$$X = \alpha(F^*\hat{\otimes}_\pi E) = X^{**}.$$

Thus, X is reflexive.

Case b) The scalar field is complex.

If A is a complex Banach space, we denote by A_r the space A considered over the real field.

If A is reflexive, so is A_r.

Let x and y^* be strongly exposed points of $U(E_r)$ and $U((F^*)_r)$ respectively. Suppose that x and y^* are exposed by $f \in (E_r)^*$ and $g \in ((F^*)_r)^*$, respectively with $\|f\| = \|g\| = 1$. Define $x^* \in E^*$ and $y \in F$ by $\langle z, x^*\rangle = f(z) - if(iz)$, for all $z \in E$, $(i^2 = -1)$ and $\langle y, z^*\rangle = g(z^*) - ig(iz^*)$, for all $z^* \in F^*$. One knows that $\|x^*\| = \|y\| = 1$.

Let $\varepsilon > 0$. There exists a number δ, $0 < \delta < 1$, such that:

$$\text{For all } \quad z \in U(E)\backslash B(x,\varepsilon) \qquad\qquad f(x - z) \geqq \delta,$$

$$\text{For all } \quad z^* \in U(F^*)\backslash B(y^*,\varepsilon), \qquad\qquad g(y^* - z^*) \geqq \delta.$$

We define v and u as in case a). There is a representation of u which has the same properties; $\lambda_i \geqq 0$, $\|x_i\| = \|y^*_i\| = 1$ and $\langle x_i, x^*\rangle$ real nonnegative.

In place of Equation (*) we now have:

$$1 = \mathrm{Re}\Big(\sum_{i=1}^{\infty} \lambda_i \langle x_i, x^*\rangle \langle y, y^*_i\rangle\Big)$$

$$= \sum_{i=1}^{\infty} \lambda_i \langle x_i, x^*\rangle \mathrm{Re}(\langle y, y^*_i\rangle),$$

$$1 = \sum_{i=1}^{\infty} \lambda_i f(x_i) g(y^*_i).$$

And it is possible to finish the proof as in case a).

COROLLARY 2.1. *Let E and F be reflexive Banach spaces such that E or F has the approximation property. If $\mathscr{C}(E,F) \neq \mathscr{L}(E,F)$, then $\mathscr{C}(E,F)$ is nonconjugate.*

PROOF. One knows (cf.A. Grothendick [4]) that in the conditions of the Corollary, $(\mathscr{C}(E,F))^{**} = \mathscr{L}(E,F)$, then, $\mathscr{C}(E,F)$ is non reflexive and the result follows from Theorem 2.

We now have a generalization of a result of R. Schatten, [11]:

COROLLARY 2.2. *Let E be a reflexive Banach space which has the approximation property and is of infinite dimension. Then $\mathscr{C}(E,E)$ is non-conjugate.*

PROOF. It is clear from Corollary 2.1.

REFERENCES

1. M. Day, *Normed Linear Spaces*, Third edition, Springer Verlag 1973.

2. J. Diestel, *The Radon-Nikodym property and the coincidence of integral and nuclear operators*, Rev. Roumaine Math. Pures Appl. **17** (1972), 1611–1620.

3. N. Dunford and J. T. Schwartz, *Linear Operators, Part I*, Interscience Pub. New York, 1958.

4. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. no. **16**, 1955.

5. A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. São Paulo **8** (1956), 1–79.

6. J. R. Holub, *Hilbertian operators and reflexive tensor products*, Pacific J. Math. **36** (1971), 185–194.

7. J. R. Holub, *Reflexivity of L(E,F)*, Proc. Amer. Math. Soc. **39** (1973), 175–177.

8. D. R. Lewis, *Conditional weak compactness in certain injective tensor products*, Math. Ann. **201** (1973), 201–209.

9. J. Lindenstrauss, *Weakly compact sets, their topological properties and the Banach spaces they generate*, Ann. of Math. Studies **69**, 1972.

10. P. Saphar, *Produits tensoriels d'espaces de Banach et classes d'applications linéaires*, Studia Math. **38** (1970), 70–100.

11. R. Schatten, *The space of completely continuous operators on a Hilbert space*, Math. Ann. **134** (1957), 47–49.

12. R. Schatten, *A theory of cross-spaces*, Ann. of Math. Studies **26**, 1950.

13. C. Stegall, *The Radon-Nikodym property in conjugate Banach spaces*, (to appear).

14. S. Troyanski, *On locally uniformly convex and differentiable norms in certain non-separable Banach spaces*, Studia Math. **37** (1970–1971), 173–180.

15. Y. Gordon, D. R. Lewis and J. R. Retherford, *Banach ideals of operators with applications*, J. Functional Analysis **14** (1973), 85–129.

16. F. E. Alexander, *Compact and finite rank operators on subspaces of l_p*, Bull. London Math. Soc. **6** (1974), 341–342.

17. N. J. Kalton, *Spaces of compact operators*, Math. Ann. **208** (1974), 267–278.

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